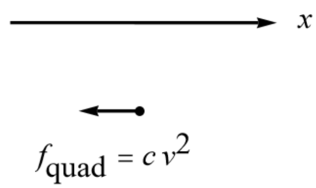
Problem 2.25

Consider the cyclist of Section 2.4, coasting to a halt under the influence of a quadratic drag force. Derive in detail the results (2.49) and (2.51) for her velocity and position, and verify that the constant $\tau = m/cv_o$ is indeed a time.

Solution

Draw a free-body diagram for a cyclist travelling to the right in a medium with quadratic air resistance.



Apply Newton's second law in the x-direction.

$$\sum F_x = ma_x$$

Let $v_x = v$ to simplify the notation.

$$-cv^2 = m\frac{dv}{dt}$$

Solve this differential equation for v by separating variables.

$$-\frac{c}{m}\,dt = \frac{dv}{v^2}$$

Integrate both sides definitely, assuming that at $t = 0$ the velocity is v_o.

$$\int_0^t -\frac{c}{m}\,dt' = \int_{v_o}^{v} \frac{dv'}{v'^2}$$

$$-\left.\frac{c}{m}t'\right|_0^t = -\left.\frac{1}{v'}\right|_{v_o}^{v}$$

$$-\frac{c}{m}(t-0) = -\left(\frac{1}{v} - \frac{1}{v_o}\right)$$

$$-\frac{c}{m}t = -\frac{1}{v} + \frac{1}{v_o}$$

$$\frac{1}{v} = \frac{ct}{m} + \frac{1}{v_o}$$

$$\frac{1}{v} = \frac{cv_o t + m}{mv_o}$$

Invert both sides to get v.

$$v(t) = \frac{mv_o}{cv_ot + m}$$

$$= \frac{mv_o}{cv_ot + m} \cdot \frac{\frac{1}{m}}{\frac{1}{m}}$$

$$= \frac{v_o}{\frac{cv_ot}{m} + 1}$$

Therefore,

$$v(t) = \frac{v_o}{1 + t/\tau}, \tag{2.49}$$

where

$$\tau = \frac{m}{cv_o}.$$

This formula for $v(t)$ is Equation (2.49) on page 59. Check to see that it has dimensions of seconds. Note that $c = \gamma D^2$ for a spherical particle, where $\gamma = 0.25\ \mathrm{N \cdot s^2/m^4}$ in air at STP.

$$[\tau] = \left[\frac{m}{cv_o}\right]$$

$$= \frac{[m]}{[c][v_o]}$$

$$= \frac{\mathrm{kg}}{\left(\frac{\mathrm{N \cdot s^2}}{\mathrm{m^4}} \cdot \mathrm{m^2}\right) \cdot \frac{\mathrm{m}}{\mathrm{s}}}$$

$$= \frac{\mathrm{kg}}{\left(\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s^2}} \cdot \frac{\mathrm{s^2}}{\mathrm{m^2}}\right) \cdot \frac{\mathrm{m}}{\mathrm{s}}}$$

$$= \frac{1}{\frac{1}{\mathrm{s}}}$$

$$= \mathrm{s}$$

Replace $v(t)$ with dx/dt in Equation (2.49).

$$\frac{dx}{dt} = \frac{v_o}{1 + t/\tau}$$

Separate variables to solve for x as before.

$$dx = \frac{v_o}{1 + t/\tau}\, dt$$

Integrate both sides definitely, assuming that at $t = 0$ the position is x_o.

$$\int_{x_o}^{x} dx' = \int_0^t \frac{v_o}{1 + t'/\tau}\, dt'$$

Evaluate the integrals on both sides.

$$
\begin{aligned}
x - x_o &= \int_0^t \frac{v_o}{1 + t'/\tau} \cdot \frac{\tau}{\tau} \, dt' \\
&= v_o \tau \int_0^t \frac{1}{\tau + t'} \, dt' \\
&= v_o \tau \ln(\tau + t') \Big|_0^t \\
&= v_o \tau [\ln(\tau + t) - \ln(\tau + 0)] \\
&= v_o \tau \ln\left(\frac{\tau + t}{\tau}\right) \\
&= v_o \tau \ln(1 + t/\tau)
\end{aligned}
$$

Setting $x_o = 0$ gives

$$x(t) = v_o \tau \ln(1 + t/\tau), \tag{2.51}$$

which is Equation (2.51) on page 59.